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Gain-Scheduled Control: Relaxing Slow Variation Requirements by Velocity-Based Design

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This paper investigates the application of the gain-scheduling approach to flight control tasks, where the conditions required by conventional techniques need not be satisfied. The conditions required by conventional techniques are progressively relaxed, and the design of gain-scheduled controllers under a range of conditions is addressed. In particular, the paper considers gain-scheduled control design in situations where the vehicle is maneuvering aggressively far from equilibrium, the airspeed need not be slowly varying, and/or scheduling on the instantaneous incidence angle is required.

Nomenclature

C_L	=	aerodynamic force coefficient
C_M	=	aerodynamic moment coefficient
c	=	characteristic distance (0.228 m)
f	=	$180/\pi$
I_{yy}	=	pitching moment of inertia (247.437 kg m^2)
M	=	pitching moment, N
$\bar{M}$	=	Mach number
$M_\alpha(\rho_0)$	=	$(1/I_{yy})(\partial M/\partial\alpha)\vert_{\rho_0}$
$M_\delta(\rho_0)$	=	$(1/I_{yy})(\partial M/\partial\delta)\vert_{\rho_0}$
m	=	mass of the missile (204.02 kg)
q	=	body axis pitch rate, rad/s
$\bar{q}$	=	dynamic pressure, $\frac{1}{2}\rho V^2$
$\boldsymbol{r}$, u, v	=	system inputs
S	=	characteristic area (0.0409 m^2)
V	=	airspeed, m/s
$\boldsymbol{x}$, $\boldsymbol{w}$, $\boldsymbol{x}^i$, $\boldsymbol{w}^i$, ω, $\boldsymbol{z}$	=	state-vectors
$\boldsymbol{y}$	=	system output
Z	=	normal force, N
$Z_\alpha(\rho_0)$	=	$(1/mV)(\partial Z/\partial\alpha)\vert_{\rho_0}$
$Z_\delta(\rho_0)$	=	$(1/mV)(\partial Z/\partial\delta)\vert_{\rho_0}$
α	=	angle of incidence, rad
Δ (and sometimes δ)	=	perturbation quantity (e.g., $\Delta q = q - q_0$)
δ	=	effective elevator deflection, rad (δ can represent the combined action of several control surfaces rather than simply the elevator alone)
ε (with various subscripts)	=	residual terms
η_z	=	normal acceleration, m/s^2
π	=	indexes equilibrium points of missile
π_0	=	specific equilibrium point
ρ	=	air density
$\boldsymbol{\rho}$	=	scheduling variable
$\boldsymbol{\rho}_0$, δ_0, q_0, α_0, η_{z0}	=	values of $\boldsymbol{\rho}$, δ, q, α, and η_z at the equilibrium point π_0
$\hat{}$	=	estimated quantity
$\dot{}$	=	d/dt

I. Introduction

GAIN-SCHEDULING control is widely employed in flight-control applications, where high performance has to be achieved over a broad operating envelope. In the gain-scheduling design approach (see, for example, Refs. 1 and 2) a nonlinear controller is constructed by continuously interpolating, in some manner, between the members of a family of linear controllers. Each linear controller is, typically, associated with a specific equilibrium operating point of the aircraft and is designed to ensure that, locally to the equilibrium operating point, the performance requirements are met. By employing a series expansion linearization that, locally to the equilibrium operating point, has similar dynamics to the aircraft, linear techniques can be used to resolve this local design task. Continuity is, therefore, maintained with established linear design techniques for which a considerable body of experience has been accumulated.

Although the traditional gain-scheduling approach just described is extremely successful in most flight-control applications,[3] conventional theoretical techniques for analyzing the dynamics of gain-scheduled systems are poorly developed and provide little support for the gain-scheduling design approach.[4] Moreover, the trend is toward aircraft and missile configurations, where the conventional gain-scheduling conditions may not always be satisfied (see, for example, Refs. 5 and 6). Gain-scheduled controllers are traditionally designed on the basis of the dynamics relative to a family of trim conditions assuming that the airspeed is slowly varying. However, during aggressive maneuvering, the vehicle may be far from equilibrium with rapidly varying airspeed (Ref. 3, p. 523). In addition, the requirement to operate at high angles of attack can necessitate scheduling on instantaneous incidence angle rather than, for example, conventional flap scheduling on averaged incidence (Ref. 3, p. 523). There is, consequently, interest in the literature in alternative nonlinear control design approaches such as dynamic inversion (see, for example, Refs. 5 and 7). However, owing to the substantial body of experience that has been accumulated with gain-scheduling methods both with regard to meeting performance requirements and also such practical issues as safety certification, there is a strong incentive to retain the gain-scheduling approach. The aim of this paper is, therefore, to investigate the application of the gain-scheduling approach to flight-control design tasks when the conditions required by conventional techniques need not be satisfied.

The velocity-based analysis and design framework, recently proposed in Refs. 4, 8, and 9, associates a linear system with every operating point of a nonlinear system, not just the equilibrium operating points. This approach thereby relaxes the restriction to near equilibrium operation while maintaining the continuity with linear methods, which is a principle advantage of the conventional gain-scheduling approach. Moreover, the velocity-based approach does not inherently involve a slow variation requirement unlike the conventional gain-scheduling approach, which requires a slow variation restriction to keep the system close to the equilibrium operating

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points. It therefore provides a natural and unified framework for gain-scheduling analysis and design, which addresses many of the shortcomings of conventional gain-scheduling analysis and design and is an appropriate framework within which to study the relaxation of the restrictions associated with the conventional gain-scheduling approach.

The paper is organized as follows. In Sec. II, a missile example is described that motivates the later discussion. The velocity-based gain-scheduling approach is briefly summarized in Sec. III and applied to the missile example in Sec. IV to study the influence on the controller of progressively weaker assumptions regarding the rate of variation of the scheduling variable. The conclusions are summarized in Sec. V.

II. Motivating Example

Consider a missile (similar to that studied by Nichols et al.[10] and Shamma and Cloutier[11]) with longitudinal dynamics

$$\dot{q} = M/I_{yy}, \qquad \dot{\alpha} = Z/mV + q, \qquad \eta_z = Z/m \tag{1}$$

with

$$Z = \bar{q}SC_L, \qquad M = \bar{q}ScC_M \tag{2}$$

Consideration is restricted to the glide phase of the missile trajectory, where the mass is constant and thrust is not applied. The aerodynamic force and moment coefficients, for Mach numbers between 1 and 4, are

$$C_L = (-0.4 + 0.033\bar{M})f\alpha - 0.034f\delta$$
$$C_M = -0.3f\alpha - 0.206f\delta \tag{3}$$

The pitch rate and the normal acceleration are measured, but the angle of incidence cannot be reliably measured. The requirement is to design a controller for the skid-to-turn missile, which achieves a uniform normal acceleration step response over the entire flight envelope with a rise time (to 95% of final value) of around 0.3 s and overshoot less than 25%. Although this is, of course, not a complete performance specification, it is adequate for the present study.

The conventional gain-scheduling design approach requires appropriate linearizations of the dynamics that approximate, locally to specific equilibrium flight conditions, the nonlinear dynamic behavior of the missile. Let the equilibrium operating points be parameterized by $\boldsymbol{\pi}$, for example, $[\delta \;\; V \;\; \bar{q} \;\; \bar{M}]^T$. Adopting the standard short-period approximation (see, for example, Ref. 3, p. 78, and Ref. 12), assume that, locally to a specific equilibrium operating point at which $\boldsymbol{\pi}$ equals $\boldsymbol{\pi}_0$, the variations in the forward velocity, dynamic pressure, and Mach number associated with the pitch motion are negligible. The nonlinear dynamics of the missile [Eq. (1)] can then be approximated, locally to the specific equilibrium operating point $\boldsymbol{\pi}_0$, by the series expansion linearization

$$\Delta\dot{q} = M_\alpha(\boldsymbol{\rho}_0)\Delta\alpha + M_\delta(\boldsymbol{\rho}_0)\Delta\delta$$
$$\Delta\dot{\alpha} = Z_\alpha(\boldsymbol{\rho}_0)\Delta\alpha + Z_\delta(\boldsymbol{\rho}_0)\Delta\delta + \Delta q \tag{4}$$

$$\Delta\eta_Z = Z_\alpha(\boldsymbol{\rho}_0)V(\boldsymbol{\rho}_0)\Delta\alpha + Z_\delta(\boldsymbol{\rho}_0)V(\boldsymbol{\rho}_0)\Delta\delta \tag{5}$$

together with the input, output, and state transformations

$$\Delta\delta = \delta - \delta_0, \qquad q = q_0 + \Delta q, \qquad \alpha = \alpha_0 + \Delta\alpha$$
$$\eta_Z = \eta_{Z_0} + \Delta\eta_Z \tag{6}$$

where $\boldsymbol{\rho} = [V \;\; \bar{q} \;\; \bar{M}]^T$ and $\boldsymbol{\rho}_0$, δ_0, q_0, α_0, and η_{Z_0} are, respectively, the values of $\boldsymbol{\rho}$, δ, q, α, and η_Z at the equilibrium operating point $\boldsymbol{\pi}_0$ and

$$M_\alpha(\boldsymbol{\rho}_0) = \frac{1}{I_{yy}}\frac{\partial M}{\partial\alpha}\bigg|_{\rho_0}, \qquad M_\delta(\boldsymbol{\rho}_0) = \frac{1}{I_{yy}}\frac{\partial M}{\partial\delta}\bigg|_{\rho_0}$$
$$Z_\alpha(\boldsymbol{\rho}_0) = \frac{1}{mV}\frac{\partial Z}{\partial\alpha}\bigg|_{\rho_0}, \qquad Z_\delta(\boldsymbol{\rho}_0) = \frac{1}{mV}\frac{\partial Z}{\partial\delta}\bigg|_{\rho_0} \tag{7}$$

Differentiating, Eq. (4) can be reformulated as the linear short period equation

$$\Delta\ddot{\alpha} - Z_\alpha(\boldsymbol{\rho}_0)\Delta\dot{\alpha} - M_\alpha(\boldsymbol{\rho}_0)\Delta\alpha = M_\delta(\boldsymbol{\rho}_0)\Delta\delta + Z_\delta(\boldsymbol{\rho}_0)\Delta\dot{\delta} \tag{8}$$

Because the magnitude of Z_δ is much less than the magnitude of M_δ (specifically, $|Z_\delta/M_\delta|$ is less than 0.003 over the entire flight envelope), Eq. (8) can, for control design purposes, be simplified to

$$\Delta\ddot{\alpha} - Z_\alpha(\boldsymbol{\rho}_0)\Delta\dot{\alpha} - M_\alpha(\boldsymbol{\rho}_0)\Delta\alpha = M_\delta(\boldsymbol{\rho}_0)\Delta\delta \tag{9}$$

which is valid locally to the specific equilibrium operating point $\boldsymbol{\pi}_0$. Hence, the linearized plant dynamics at the equilibrium operating point $\boldsymbol{\pi}_0$ are described by Eqs. (9) and (5) together with the input, output, and state transformations [Eqs. (6)]. Although the input, output, and state transformations [Eqs. (6)] are different at every equilibrium operating point, the linearized dynamics [Eqs. (9) and (5)] are the same at equilibrium operating points for which $\boldsymbol{\rho}$ equals $\boldsymbol{\rho}_0$. Hence Eqs. (9) and (5) define a linear dynamic family parameterized by $\boldsymbol{\rho}$.

For each member of the linear family, Eqs. (9) and (5), a linear controller is designed to meet the performance specification. A cascaded inner-outer loop controller configuration is employed with an attitude inner loop and a normal acceleration outer loop (see Appendix A for details). The resulting linear controller family, expressed in state-space form, is

$$\begin{bmatrix}\Delta\dot{\hat{\alpha}}\\ \Delta\dot{x}_1\\ \Delta\dot{x}_2\\ \Delta\dot{x}_3\\ \Delta\dot{x}_4\end{bmatrix} = \begin{bmatrix} Z_\alpha(\boldsymbol{\rho}_0) - Z_\delta(\boldsymbol{\rho}_0)\dfrac{K_1}{M_\delta(\boldsymbol{\rho}_0)} & Z_\delta(\boldsymbol{\rho}_0) & 0 & 0 & Z_\delta(\boldsymbol{\rho}_0)\dfrac{K_1}{M_\delta(\boldsymbol{\rho}_0)}\dfrac{w_n^2}{Z_\alpha(\boldsymbol{\rho}_0)V(\boldsymbol{\rho}_0)} \\ [K_2 + Z_\alpha(\boldsymbol{\rho}_0)]\dfrac{K_1}{M_\delta(\boldsymbol{\rho}_0)} & -K_2 & 1 & 0 & -[K_2 + Z_\alpha(\boldsymbol{\rho}_0)]\dfrac{K_1}{M_\delta(\boldsymbol{\rho}_0)}\dfrac{w_n^2}{Z_\alpha(\boldsymbol{\rho}_0)V(\boldsymbol{\rho}_0)} \\ M_\alpha(\boldsymbol{\rho}_0)\dfrac{K_1}{M_\delta(\boldsymbol{\rho}_0)} & 0 & 0 & 0 & -M_\alpha(\boldsymbol{\rho}_0)\dfrac{K_1}{M_\delta(\boldsymbol{\rho}_0)}\dfrac{w_n^2}{Z_\alpha(\boldsymbol{\rho}_0)V(\boldsymbol{\rho}_0)} \\ 0 & 0 & 0 & -2\varsigma w_n & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix}\Delta\hat{\alpha}\\ \Delta x_1\\ \Delta x_2\\ \Delta x_3\\ \Delta x_4\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}\Delta q\\ \Delta\eta_{Z_d} - \Delta\eta_Z\end{bmatrix} \tag{10}$$

$$\Delta\delta = \begin{bmatrix} -\dfrac{K_1}{M_\delta(\boldsymbol{\rho}_0)}\dfrac{w_n^2}{Z_\alpha(\boldsymbol{\rho}_0)V(\boldsymbol{\rho}_0)} & 1 & 0 & 0 & \dfrac{K_1}{M_\delta(\boldsymbol{\rho}_0)}\dfrac{w_n^2}{Z_\alpha(\boldsymbol{\rho}_0)V(\boldsymbol{\rho}_0)} \end{bmatrix}\begin{bmatrix}\Delta\hat{\alpha}\\ \Delta x_1\\ \Delta x_2\\ \Delta x_3\\ \Delta x_4\end{bmatrix} \tag{11}$$

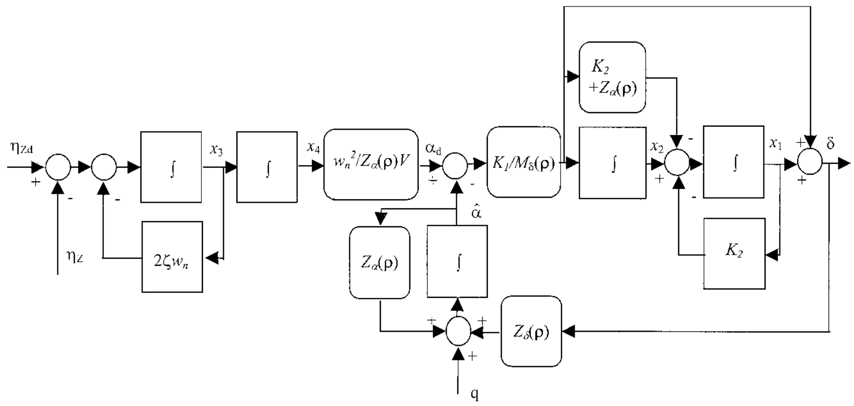

Fig. 1 Nonlinear controller realization A.

where $\Delta\eta_{Z_d}$ is the perturbation in the demanded normal acceleration and $K_1 = 2500$, $K_2 = 70$, $w_n = 10$ rad/s, and $\zeta = 0.7$. Because the controllers are designed on the basis of the linearization of the missile dynamics at trim flight conditions, the linear controllers act on perturbed quantities. The input, output, and state transformations relating the perturbed quantities to their measured counterparts depend on the specific equilibrium flight condition considered. Hence, a family of input, output, and state transformations, parameterized by the equilibrium flight condition, is also associated with the family of linear controller transfer functions.

The family of linear controllers is parameterized by ρ, that is, by the airspeed, dynamic pressure, and Mach number of the missile. Perhaps the most widespread approach to constructing a nonlinear controller realization is to simply substitute these scheduling variables into the linear controller family [Eqs. (10) and (11)] that is, simply vary the gains of the controller according to the airspeed, dynamic pressure, and Mach number (see, for example, Ref. 3, p. 298, and Refs. 10 and 12) while relying on the feedback action to accommodate the input, output, and state transformations. The resulting nonlinear controller, denoted controller realization A, is depicted in Fig. 1. The response, obtained using nonlinear simulations, of the nonlinear controller to a sequence of step demands in normal acceleration is shown in Fig. 2 (the Mach number time history, shown in Fig. 3, is employed. Only a very simple representation of the missile is employed, and, in particular, the translational dynamics are not modeled. The airspeed, dynamic pressure, and Mach-number time histories are generated by auxiliary equations in a manner similar to Ref. 10. Nevertheless, the model is sufficient to illustrate the dependence of the closed-loop dynamics on the rate of variation of these quantities). Evidently, the nonlinear controller achieves the performance specification of a uniform normal acceleration step response over the flight envelope with rise time of around 0.3 s and overshoot less than 25%.

The conventional gain-scheduling approach followed in the preceding discussion does not uniquely define the nonlinear controller. State-space representations of a linear controller, which are related by a nonsingular linear state transformation, are linearly equivalent, that is, they have the same transfer functions. Hence, each member of the family of linear controllers [Eqs. (10) and (11)] has infinitely many state-space representations. For example, an alternative controller family is

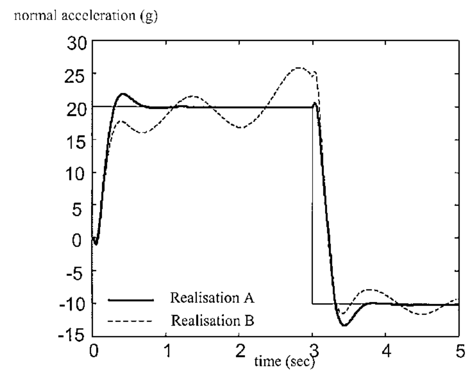

Fig. 2 Performance of nonlinear controllers.

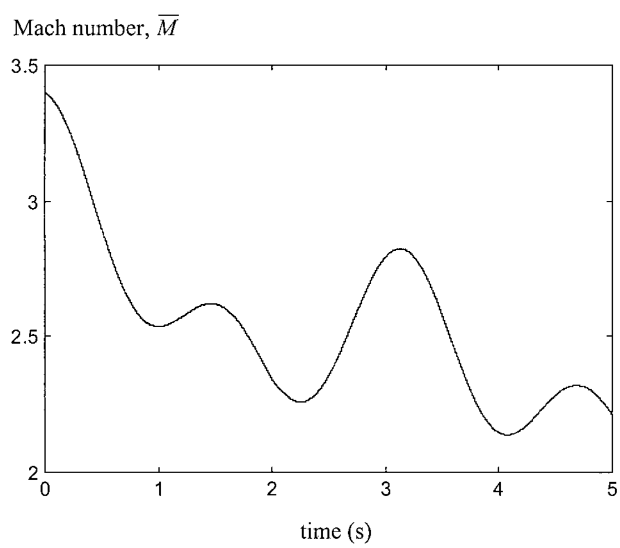

Fig. 3 Mach-number time history.

$$
\begin{bmatrix} \Delta\dot{\hat{\alpha}} \\ \Delta\dot{\tilde{x}}_1 \\ \Delta\dot{\tilde{x}}_2 \\ \Delta\dot{\tilde{x}}_3 \\ \Delta\dot{\tilde{x}}_4 \end{bmatrix} = \begin{bmatrix} Z_\alpha(\rho_0) - Z_\delta(\rho_0)\dfrac{K_1}{M_\delta(\rho_0)} & Z_\delta(\rho_0) & 0 & 0 & Z_\delta(\rho_0)\dfrac{K_1}{M_\delta(\rho_0)} \\ [K_2 + Z_\alpha(\rho_0)]\dfrac{K_1}{M_\delta(\rho_0)} & -K_2 & 1 & 0 & -[K_2 + Z_\alpha(\rho_0)]\dfrac{K_1}{M_\delta(\rho_0)} \\ M_\alpha(\rho_0)\dfrac{K_1}{M_\delta(\rho_0)} & 0 & 0 & 0 & -M_\alpha(\rho_0)\dfrac{K_1}{M_\delta(\rho_0)} \\ 0 & 0 & 0 & -2\varsigma w_n & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta\hat{\alpha} \\ \Delta\tilde{x}_1 \\ \Delta\tilde{x}_2 \\ \Delta\tilde{x}_3 \\ \Delta\tilde{x}_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \dfrac{w_n^2}{Z_\alpha(\rho_0)V(\rho_0)} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta q \\ \Delta\eta_{Z_d} - \Delta\eta_Z \end{bmatrix} \tag{12}
$$

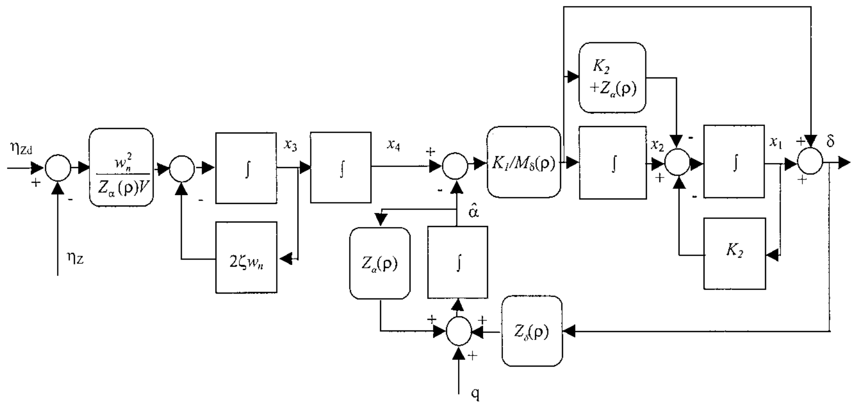

Fig. 4 Nonlinear controller realization B.

$$\Delta\delta = \begin{bmatrix} -\dfrac{K_1}{M_\delta(\rho_0)} & 1 & 0 & 0 & \dfrac{K_1}{M_\delta(\rho_0)} \end{bmatrix} \begin{bmatrix} \Delta\hat{\alpha} \\ \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_d \end{bmatrix} \tag{13}$$

The linear family, [Eqs. (12) and (13)], employs a different choice of state from Eqs. (10) and (11). However, it is emphasized that the members of the families of linear controllers defined by Eqs .(10) and (11) and Eqs. (12) and (13) are dynamically equivalent, that is, they have the same transfer functions. In this example the linear family, [Eqs. (12) and (13)], is obtained by simply altering the location of controller outer-loop gain, which in realization A is scheduled. The nonlinear controller realization constructed by directly scheduling the controller gains of the linear family [Eqs. (12) and (13)] with respect to airspeed, dynamic pressure, and Mach number, denoted controller realization B, is depicted in Fig. 4. The response of this nonlinear controller to a sequence of step demands in normal acceleration is also shown in Fig. 2. Evidently, the performance of nonlinear controller realizations A and B is markedly different. Although the nonlinear controller with realization A achieves the performance specification, the response with the nonlinear controller with realization B fails to satisfy the rise time and overshoot specification. Indeed, with this controller realization the response is quite oscillatory and, perhaps, divergent.

Clearly, the performance attained in this example is sensitive to the choice of controller realization. Because the controllers are different realizations for the same family of linear controller transfer functions, the dynamic behavior, and consequently the closed-loop performance, of the controllers is similar when the scheduling variable varies sufficiently slowly (trivially, when the rate of change is zero). However, because the airspeed, dynamic pressure, and Mach number are directly related to the normal acceleration of the missile, the variations in the scheduling variables are not a priori slowly varying relative to the dynamics of the controller, in particular, with respect to the dynamics of the acceleration outer-loop controller (as demonstrated by the difference in the response of realization A and realization B). Moreover, although the controllers are designed on the basis of the missile dynamics relative to trim conditions, the missile motion in this example is not confined to small perturbations about trim but rather involves aggressive maneuvering, which takes the missile far from equilibrium. The assumptions underlying the conventional gain-scheduling procedure are clearly not met. Nevertheless, the controller realization A does meet the performance specification. Hence, the potential clearly exists to extend the gain-scheduling design methodology to accommodate situations, such as the present example, where the conventional approach is no longer warranted.

The selection of an appropriate gain-scheduling controller realization is considered by Refs. 13–15. The utility of the former approach is, however, somewhat limited in general.[16,17] In the latter approach, a controller realization is sought, which leads to the weakest slow variation requirement within the context of conventional gain scheduling, that is, on the basis of the plant dynamics relative to the equilibrium operating points. However, the slow variation requirement can, in general, be further weakened by exploiting knowledge of the plant dynamics at nonequilibrium operating points.[8] In the present missile example the controller realizations do not belong to either of the classes studied by Refs. 13–15.

III. Velocity-Based Gain Scheduling

Despite the widespread application of gain-scheduled controllers, conventional theoretical techniques for analyzing the dynamics of gain-scheduled systems are rather poorly developed. A detailed review is presented in Ref. 4. Because the conventional gain-scheduling design approach is based on combining linear controllers designed on the basis of plant linearizations about a number of equilibrium operating points, the resulting nonlinear controller is only appropriate, in general, in the vicinity of the equilibrium operating points. This, in turn, implies an inherent slow variation requirement to keep the system trajectories close to the equilibrium operating points. Nonetheless, the stability of the closed-loop dynamics, locally to a specific equilibrium operating point, can be analyzed by series expansion linearization theory. In addition, provided the family of equilibrium operating points can be parameterized by the input to the closed-loop system, frozen-input theory can be employed to analyze certain dynamic properties near the family of equilibrium operating points. However, the latter analysis is restricted to a small neighborhood of the equilibrium operating points. Furthermore, it is based on a frozen-input representation of the controlled system, which is quite distinct from the mixed series-expansion/frozen-scheduling variable representation employed in the conventional gain-scheduling design procedure. (In the conventional gain-scheduling approach a nonlinear controller is obtained by combining, in some manner, the members of a family of linear controllers. A typical procedure is simply to let the scheduling variable vary over the linear controller family while neglecting the input, output, and state transformations relating the input, output, and state of the nonlinear plant and controller to the perturbed quantities with respect to which the members of the linear plant and controller families are described. Hence, although the series expansion linearizations of the plant are employed in the design procedure, the corresponding local controller designs are frozen-scheduling variable linearizations of the resulting nonlinear controller.) Similarly, although the frozen-scheduling variable linearization is employed during the design procedure, the series expansion linearization of the controller is employed when analyzing the dynamic behavior locally to a single equilibrium operating point. Hence, the analysis of the dynamic properties of the controlled system near a specific equilibrium operating point does not reduce to either the series expansion analysis or the mixed series-expansion/frozen-scheduling variable analysis employed in the design procedure. The foregoing analysis tools therefore provide limited support for the conventional gain-scheduling design approach and are unsuitable for investigating the application of gain scheduling to flight control tasks when the aircraft motion may take it far from equilibrium and the scheduling variable may not be slowly varying.

The velocity-based analysis and design framework, proposed recently in Refs. 4, 8, and 9, associates a linearization with every operating point of a nonlinear system, not just the equilibrium operating points. The approach thereby relaxes the restriction to near-equilibrium operation while maintaining the continuity with linear methods, which is a principle advantage of conventional gain scheduling. Moreover, the velocity-based approach does not inherently involve a slow variation requirement. It therefore provides an appropriate framework within which to investigate the relaxation of the restrictions associated with conventional gain scheduling. In contrast to the conventional analysis methods just discussed, the velocity-based framework provides a unified framework for gain-scheduling analysis and design, which uses a single type of linearization, namely, the velocity-based linearization, for both analysis and design. It is briefly summarised next.

Consider the nonlinear system

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{r}), \qquad \boldsymbol{y} = \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{r}) \tag{14}$$

where $\boldsymbol{F}(\cdot, \cdot)$ and $\boldsymbol{G}(\cdot, \cdot)$ are differentiable with bounded, Lipschitz continuous derivatives. The set of equilibrium operating points of the nonlinear system [Eq. (14)] consists of those points $(\boldsymbol{x}_0, \boldsymbol{r}_0)$ for which

$$\boldsymbol{F}(\boldsymbol{x}_0, \boldsymbol{r}_0) = 0 \tag{15}$$

Let $\Phi\colon \Re^n \times \Re^m$ denote the space consisting of the product of the state $\boldsymbol{x}$ with the input $\boldsymbol{r}$. The set of equilibrium operating points of the nonlinear system [Eq. 14)] forms a locus of points $(\boldsymbol{x}_0, \boldsymbol{r}_0)$ in Φ, and the response of the system to a general time-varying input $\boldsymbol{r}(t)$ is depicted by a trajectory in Φ.

Suppose that the nonlinear system (14) is evolving along a trajectory $[\boldsymbol{x}(t), \boldsymbol{r}(t)]$ in Φ and at time t_1; the trajectory has reached the point $(\boldsymbol{x}_1, \boldsymbol{r}_1)$. From Taylor-series expansion theory the subsequent behavior of the nonlinear system can be approximated, locally to the point, $(\boldsymbol{x}_1, \boldsymbol{r}_1)$ by the first-order representation

$$\delta\dot{\hat{\boldsymbol{x}}} = \boldsymbol{F}(\boldsymbol{x}_1, \boldsymbol{r}_1) + \nabla_x \boldsymbol{F}(\boldsymbol{x}_1, \boldsymbol{r}_1)\delta\hat{\boldsymbol{x}} + \nabla_r \boldsymbol{F}(\boldsymbol{x}_1, \boldsymbol{r}_1)\delta\boldsymbol{r} \tag{16}$$

$$\delta\hat{\boldsymbol{y}} = \nabla_x \boldsymbol{G}(\boldsymbol{x}_1, \boldsymbol{r}_1)\delta\hat{\boldsymbol{x}} + \nabla_r \boldsymbol{G}(\boldsymbol{x}_1, \boldsymbol{r}_1)\delta\boldsymbol{r} \tag{17}$$

$$\delta\boldsymbol{r} = \boldsymbol{r} - \boldsymbol{r}_1, \qquad \hat{\boldsymbol{y}} = \boldsymbol{G}(\boldsymbol{x}_1, \boldsymbol{r}_1) + \delta\hat{\boldsymbol{y}}, \qquad \hat{\boldsymbol{x}} = \delta\hat{\boldsymbol{x}} + \boldsymbol{x}_1, \qquad \dot{\hat{\boldsymbol{x}}} = \delta\dot{\hat{\boldsymbol{x}}} \tag{18}$$

provided $\boldsymbol{x}_1 + \delta\hat{\boldsymbol{x}} \subseteq N_x\, \boldsymbol{r}_1 + \delta\boldsymbol{r} \subseteq N_r$, where the neighborhoods N_x and N_r of, respectively, $\boldsymbol{x}_1$ and $\boldsymbol{r}_1$ are sufficiently small. The expansion is carried out relative to the fixed operating point $(\boldsymbol{x}_1, \boldsymbol{r}_1)$ as opposed to a trajectory passing through $(\boldsymbol{x}_1, \boldsymbol{r}_1)$, and the derivative of $\boldsymbol{x}_1$ is, therefore, zero. Equations (16–18) is a well-defined system in its own right, distinct from the nonlinear system (14), with state $\delta\hat{\boldsymbol{x}}$. Furthermore, the point $(\boldsymbol{x}_1, \boldsymbol{r}_1)$ need not be an equilibrium operating point and, indeed, can lie far from the locus of equilibrium operating points. When $\delta\hat{\boldsymbol{x}}(t_1)$ is zero, the solution $\hat{\boldsymbol{x}}(t)$ to Eqs. (16–18), initially at time t_1, is tangential to the solution $\boldsymbol{x}(t)$ of Eq. (14). Indeed, locally to time t_1, $\hat{\boldsymbol{x}}(t)$ provides a second-order approximation to $\boldsymbol{x}(t)$, and $\hat{\boldsymbol{y}}(t)$ provides a first-order approximation to $\boldsymbol{y}(t)$ (Ref. 4).

The solution $\hat{\boldsymbol{x}}(t)$ to the first-order series expansion [Eqs. (16–18)], provides a valid approximation only while the solution $\boldsymbol{x}(t)$ to the nonlinear system remains in the vicinity of the operating point $(\boldsymbol{x}_1, \boldsymbol{r}_1)$. However, the solution $\boldsymbol{x}(t)$ to the nonlinear system need not stay in the vicinity of a single operating point. Consider the time interval $[0, T]$; the initial time can, without loss of generality, always be taken as zero. An approximation to $\boldsymbol{x}(t)$ is obtained by partitioning the interval into a number of short subintervals. Over each subinterval the approximate solution is the solution to the first-order series expansion relative to the operating point reached at the initial time for the subinterval (with the initial conditions for each subinterval chosen to ensure continuity of the approximate solution). The number of local solutions employed is dependent on the duration of the subintervals, but the local solutions are accurate to second order, that is, the approximation error is proportional to the duration of the subinterval cubed. Hence, as the number of subintervals increases, the approximation error associated with each rapidly decreases, and the overall approximation error also decreases. Indeed, the overall approximation error tends to zero as the maximum size of the subintervals tends to zero.[4] Hence, the family of first-order series expansions, with members defined by Eqs. (16–18), can provide an arbitrarily accurate approximation to the solution of the nonlinear system. Moreover, this approximation property holds throughout Φ and is not confined to the vicinity of a single equilibrium operating point or even of the locus of equilibrium operating points.

However, the state, input, and output transformations [Eq. (18)] depend on the operating point relative to which the series expansion is carried out. When the solution to the nonlinear system is confined to a neighborhood about a single operating point, the transformations [Eq. (18)] are static, and the dynamic behavior is described by the system (16) and (17) alone. However, when the solution to the nonlinear system traces a trajectory that is not confined to a neighborhood about a single operating point, the transformations [Eq. (18)] are no longer static, and the dynamic behavior is no longer described solely by the system (16) and (17). Instead, the dynamic behavior is described by Eqs. (16–18). Combining Eqs. (16) and (17) with the local input, output, and state transformations [Eq. (18)], each member [Eqs. (16–18)] of the family of first-order representations can be reformulated as

$$\dot{\hat{\boldsymbol{x}}} = [\boldsymbol{F}(\boldsymbol{x}_1, \boldsymbol{r}_1) - \nabla_x \boldsymbol{F}(\boldsymbol{x}_1, \boldsymbol{r}_1)\boldsymbol{x}_1 - \nabla_r \boldsymbol{F}(\boldsymbol{x}_1, \boldsymbol{r}_1)\boldsymbol{r}_1] + \nabla_x \boldsymbol{F}(\boldsymbol{x}_1, \boldsymbol{r}_1)\hat{\boldsymbol{x}} + \nabla_r \boldsymbol{F}(\boldsymbol{x}_1, \boldsymbol{r}_1)\boldsymbol{r} \tag{19}$$

$$\hat{\boldsymbol{y}} = [\boldsymbol{G}(\boldsymbol{x}_1, \boldsymbol{r}_1) - \nabla_x \boldsymbol{G}(\boldsymbol{x}_1, \boldsymbol{r}_1)\boldsymbol{x}_1 - \nabla_r \boldsymbol{G}(\boldsymbol{x}_1, \boldsymbol{r}_1)\boldsymbol{r}_1] + \nabla_x \boldsymbol{G}(\boldsymbol{x}_1, \boldsymbol{r}_1)\hat{\boldsymbol{x}} + \nabla_r \boldsymbol{G}(\boldsymbol{x}_1, \boldsymbol{r}_1)\boldsymbol{r} \tag{20}$$

In contrast to the representation [Eqs. (16) and (17)], the state, input, and output are now the same for all members of the reformulated family. The dynamics [Eqs. (19) and (20)] of an individual member of the family are affine rather than linear even when $(\boldsymbol{x}_1, \boldsymbol{r}_1)$ is an equilibrium operating point. The inhomogeneous terms in Eqs. (19) and (20) can, in general, be extremely large and can dominate the solution.

On differentiating Eqs. (19) and (20),

$$\dot{\hat{\boldsymbol{x}}} = \hat{\boldsymbol{w}} \tag{21}$$

$$\dot{\hat{\boldsymbol{w}}} = \nabla_x \boldsymbol{F}(\boldsymbol{x}_1, \boldsymbol{r}_1)\hat{\boldsymbol{w}} + \nabla_r \boldsymbol{F}(\boldsymbol{x}_1, \boldsymbol{r}_1)\dot{\boldsymbol{r}} \tag{22}$$

$$\dot{\hat{\boldsymbol{y}}} = \nabla_x \boldsymbol{G}(\boldsymbol{x}_1, \boldsymbol{r}_1)\hat{\boldsymbol{w}} + \nabla_r \boldsymbol{G}(\boldsymbol{x}_1, \boldsymbol{r}_1)\dot{\boldsymbol{r}} \tag{23}$$

The system [Eqs. (21) and (23)] is dynamically equivalent to the system [Eqs. (19) and (20)], in the sense that with appropriate initial conditions, namely,

$$\hat{\boldsymbol{x}}(t_1) = \boldsymbol{x}_1, \qquad \hat{\boldsymbol{w}}(t_1) = \boldsymbol{F}(\boldsymbol{x}_1, \boldsymbol{r}_1), \qquad \hat{\boldsymbol{y}}(t_1) = \boldsymbol{G}(\boldsymbol{x}_1, \boldsymbol{r}_1) \tag{24}$$

the solution $\hat{\boldsymbol{x}}$ to Eqs. (21–23) is the same as the solution $\hat{\boldsymbol{x}}$ to Eqs. (19) and (20). However, in contrast to Eqs. (19) and (20), the transformed system [Eqs. (21–23)] is linear. The relationship between the nonlinear system and its velocity-based linearization [Eqs. (21–23)] is direct. Differentiating Eq. (14), an alternative representation of the nonlinear system is

$$\dot{\boldsymbol{x}} = \boldsymbol{w} \tag{25}$$

$$\dot{\boldsymbol{w}} = \nabla_x \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{r})\boldsymbol{w} + \nabla_r \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{r})\dot{\boldsymbol{r}} \tag{26}$$

$$\dot{\boldsymbol{y}} = \nabla_x \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{r})\boldsymbol{w} + \nabla_r \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{r})\dot{\boldsymbol{r}} \tag{27}$$

Dynamically Eqs. (25–27), with appropriate initial conditions corresponding to Eqs. (24), and Eq. (14) are equivalent (have the same solution $\boldsymbol{x}$). [When $\boldsymbol{w} = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{r})$, $\boldsymbol{y} = \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{r})$ is invertible so that $\boldsymbol{x}$

can be expressed as a function of $\boldsymbol{w}$, $\boldsymbol{r}$ and $\boldsymbol{y}$, then the transformation relating Eqs. (25–27) to Eq. (14) is, in fact algebraic.] When

$$\hat{\boldsymbol{x}}(t_1) = \boldsymbol{x}(t_1), \qquad \hat{\boldsymbol{w}}(t_1) = \boldsymbol{w}(t_1), \qquad \hat{\boldsymbol{y}}(t_1) = \boldsymbol{y}(t_1) \tag{28}$$

$\hat{\boldsymbol{x}}(t)$ and $\hat{\boldsymbol{y}}(t)$ still provide a second- and first-order approximation to, respectively, $\boldsymbol{x}(t)$, $\boldsymbol{y}(t)$, and $\hat{\boldsymbol{w}}(t)$ provides a first-order approximation to $\boldsymbol{w}(t)$ (Ref. 4). Clearly, the velocity-based linearization [Eqs. (21–23)], is simply the frozen form of Eqs. (25–27) at the operating point $(\boldsymbol{x}_1, \boldsymbol{r}_1)$.

There exists a velocity-based linearization [Eqs. (21–23)] for every point in Φ. Hence, a velocity-based linearization family, with members defined by Eqs. (21–23), can be associated with the nonlinear system, Eq. (14). Similarly to the family of first-order expansions, the solutions to the members of the family of velocity-based linearizations [Eqs. (21–23)] can be pieced together (with the initial conditions for each subinterval chosen to ensure continuity of $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{w}}$, and $\hat{\boldsymbol{y}}$) to approximate the solution to the nonlinear system (25–27) to an arbitrary degree of accuracy.[4] Hence, the velocity-based linearization family embodies the entire dynamics of the nonlinear system [Eq. (14)], with no loss of information and provides an alternative representation of the nonlinear system. Although the velocity-based representation is equivalent to the direct representation [Eq. (14)] in the sense that they each embody the entire dynamics of the nonlinear system, they are not necessarily equivalent with respect to other considerations. In particular, the direct relationship between the velocity form of the nonlinear system and the velocity-based linearization family and the linearity of the members of the latter family provides continuity with established linear theory, which, for example, facilitates analysis (Ref. 4) and controller design.[5,6]

With regard to controller design, the velocity-based linearization of the feedback combination of a plant and controller is simply the feedback combination of the velocity-based linearizations of the plant and controller.[5] The following velocity-based gain-scheduling design procedure is, therefore, appropriate:

1) Determine the velocity-based linearization family associated with the nonlinear plant dynamics.

2) Based on the velocity-based linearization family of the plant, determine the required velocity-based linearization family of the controller such that the resulting closed-loop family achieves the performance requirements. Because each member of the plant family is linear, conventional linear design methods can be used to design each corresponding member of the controller family. Of course, it is necessary to ensure that the members of the controller family are, in some appropriate sense, compatible with one another.

3) Implement a nonlinear controller with the velocity-based linearization family designed at step 2. This step is discussed in detail in Refs. 4 and 5. When the controller contains integral action, the controller velocity-based linearization family can be implemented directly in the velocity-based form [Eqs. (25–27)].

This design procedure retains a divide and conquer approach and maintains the continuity with linear design methods, which is an important feature of the conventional gain-scheduling approach. However, in contrast to the conventional gain-scheduling approach, the resulting nonlinear controller is valid throughout the operating envelope of the plant, not just in the vicinity of the equilibrium operating points. This extension is a direct consequence of employing the velocity-based linearization framework rather than the conventional series expansion linearization about an equilibrium operating point.

Stability of the nonlinear closed-loop system is guaranteed provided the members of the closed-loop velocity-based linearization family are uniformly stable; the mapping from $\boldsymbol{w}$ to $\boldsymbol{x}$ is bounded [e.g., when $\nabla_x \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{r})$ is integrable and the integral uniformly invertible], and the class of inputs and initial conditions is restricted to limit the rate of evolution of the solution trajectories to be sufficiently small.[4] In addition, provided that the rate of evolution is sufficiently slow, the nonlinear system inherits the stability robustness of the members of the velocity-based linearization family. Nevertheless, the velocity-based gain-scheduling methodology does not inherently involve a slow variation requirement, rather, a number of factors that are a function of the controller design and that the designer is free to adjust contribute toward any limitation on the rate of evolution of the trajectories. A primary factor that contributes toward any slow variation requirement is the degree of similarity between the dynamics of the velocity-based linearizations, of the closed-loop system, associated with different operating points. This determines the degree of nonlinearity of the closed-loop system and is largely dependent on the performance specification. A secondary factor is the choice of controller realization, which, in contrast to the foregoing, is often subject to relatively few restrictions. Although the transfer function of each member of the controller velocity-based linearization family is unchanged by a nonsingular state transformation, which is different for each member of the family, the dynamics of the nonlinear controller are changed. This issue is largely neglected in the conventional gain-scheduling approach and can also be ignored in the velocity-based gain-scheduling approach provided the rate of variation of the solution trajectories is sufficiently slow because the controller dynamics are then insensitive to the choice of state (Appendix B). However, by choosing the controller realization appropriately, any limitation on the rate of evolution of the trajectories can be relaxed. When, for example, the controller is designed such that closed-loop velocity-based linearizations have the same input-output dynamics at every operating point and compatible states, there is no restriction whatsoever on the rate of evolution, and the gain-scheduled controller can be interpreted as a dynamic inversion controller.[9] Of course, the velocity-based gain-scheduling approach is not confined to the design of dynamic inversion controllers. Because a velocity-based linearization is associated with every operating point, the controller can instead be designed such that the closed-loop velocity-based linearizations vary, in some appropriate manner, across the operating envelope, for example, in the context of flight control, a uniform response is often undesirable in piloted aircraft and a degradation in the response to warn of the onset of stall can be preferred.

IV. Velocity-Based Analysis of Missile Example

By differentiating, the nonlinear missile dynamics [Eq. (1)] can be reformulated in the velocity-based form

$$\dot{\boldsymbol{x}} = \boldsymbol{w} \tag{29}$$

$$\dot{\boldsymbol{w}} = \begin{bmatrix} 0 & \dfrac{1}{I_{yy}}\dfrac{\partial M}{\partial \alpha}(\boldsymbol{\rho}) \\ 1 & \dfrac{1}{mV(\boldsymbol{\rho})}\dfrac{\partial Z}{\partial \alpha}(\boldsymbol{\rho}) \end{bmatrix} \boldsymbol{w} + \begin{bmatrix} \dfrac{1}{I_{yy}}\dfrac{\partial M}{\partial \bar{q}}(\boldsymbol{\rho}) & \dfrac{1}{I_{yy}}\dfrac{\partial M}{\partial \bar{M}}(\boldsymbol{\rho}) & 0 & \dfrac{1}{I_{yy}}\dfrac{\partial M}{\partial \delta}(\boldsymbol{\rho}) \\ \dfrac{1}{mV(\boldsymbol{\rho})}\dfrac{\partial Z}{\partial \bar{q}}(\boldsymbol{\rho}) & \dfrac{1}{mV(\boldsymbol{\rho})}\dfrac{\partial Z}{\partial \bar{M}}(\boldsymbol{\rho}) & -\dfrac{Z}{mV^2(\boldsymbol{\rho})} & \dfrac{1}{mV(\boldsymbol{\rho})}\dfrac{\partial Z}{\partial \delta}(\boldsymbol{\rho}) \end{bmatrix} \dot{\boldsymbol{r}} \tag{30}$$

$$\dot{\eta}_z = \begin{bmatrix} 0 & \dfrac{1}{m}\dfrac{\partial Z}{\partial \alpha}(\boldsymbol{\rho}) \end{bmatrix} \boldsymbol{w} + \begin{bmatrix} \dfrac{1}{m}\dfrac{\partial Z}{\partial \bar{q}}(\boldsymbol{\rho}) & \dfrac{1}{m}\dfrac{\partial Z}{\partial \bar{M}}(\boldsymbol{\rho}) & 0 & \dfrac{1}{m}\dfrac{\partial Z}{\partial \delta}(\boldsymbol{\rho}) \end{bmatrix} \dot{\boldsymbol{r}} \tag{31}$$

where

$$\boldsymbol{x} = [q \quad \alpha]^T, \qquad \boldsymbol{w} = [\dot{q} \quad \dot{\alpha}]^T, \qquad \boldsymbol{r} = [\bar{q} \quad \bar{M} \quad V \quad \delta]^T$$

A. Classical Situation

Before considering more general situations, the conventional gain-scheduling design approach is analyzed within the velocity-based framework. Assume that the aerodynamic moment and force are linear with respect to incidence and elevator angle and that $\partial M/\partial\alpha$, $\partial M/\partial\delta$, $\partial Z/\partial\alpha$, and $\partial Z/\partial\delta$ do not depend on the angle of incidence α or the elevator angle δ, that is,

$$M = \frac{\partial M}{\partial\alpha}(\rho)\alpha + \frac{\partial M}{\partial\delta}(\rho)\delta + M_\varepsilon(\rho)$$

$$Z = \frac{\partial Z}{\partial\alpha}(\rho)\alpha + \frac{\partial Z}{\partial\delta}(\rho)\delta + Z_\varepsilon(\rho) \tag{32}$$

where the scheduling variable ρ does not depend on the incidence and elevator angles. This assumption is clearly satisfied in the present missile example, in which the elements of the scheduling variable ρ are airspeed, dynamic pressure, and Mach number, and is almost always satisfied for conventional aircraft and missile configurations in normal operation out of stall (see, for example, Ref. 3, p. 103).

Under these conditions the missile dynamics [Eqs. (29–31)] can be reformulated as

$$\dot{x} = w, \qquad \dot{w} = \begin{bmatrix} 0 & M_\alpha(\rho) \\ 1 & Z_\alpha(\rho) \end{bmatrix} w + \begin{bmatrix} M_\delta(\rho) \\ Z_\delta(\rho) \end{bmatrix} \dot{\delta} + \varepsilon_w$$

$$\dot{\eta}_Z = [0 \quad Z_\alpha(\rho)V(\rho)]w + Z_\delta(\rho)V(\rho)\dot{\delta} + \varepsilon_\eta \tag{33}$$

where

$$\varepsilon_w = \begin{bmatrix} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{M}{I_{yy}}\right) - M_\alpha(\rho)\dot{\alpha} - M_\delta(\rho)\dot{\delta} \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{Z}{mV}\right) - Z_\alpha(\rho)\dot{\alpha} - Z_\delta(\rho)\dot{\delta} \end{bmatrix}$$

$$\varepsilon_\eta = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{Z}{m}\right) - Z_\alpha(\rho)V(\rho)\dot{\alpha} - Z_\delta(\rho)V(\rho)\dot{\delta} \tag{34}$$

and $M_\alpha(\rho)$, $M_\delta(\rho)$, $Z_\alpha(\rho)$, and $Z_\delta(\rho)$ are obtained by allowing the scheduling variable ρ to vary in Eqs. (7). Assume that the scheduling variable ρ is sufficiently slowly varying that ε_w and ε_η can be neglected in Eqs. (33), that is, the nonlinear missile dynamics are described by

$$\dot{x} = w, \qquad \dot{w} = \begin{bmatrix} 0 & M_\alpha(\rho) \\ 1 & Z_\alpha(\rho) \end{bmatrix} w + \begin{bmatrix} M_\delta(\rho) \\ Z_\delta(\rho) \end{bmatrix} \dot{\delta}$$

$$\dot{\eta}_Z = [0 \quad Z_\alpha V]w + Z_\delta V\dot{\delta} \tag{35}$$

The members of the family of velocity-based linearizations associated with the missile dynamics are simply the frozen forms of Eqs. (35). The assumption that ε_w and ε_η can be neglected is essentially a requirement that the short period approximation is accurate.

In the velocity-based gain-scheduling approach a family of linear controllers is designed corresponding to the family of plant velocity-based linearizations. The velocity-based linearization family includes linearizations of the plant at both nonequilibrium and equilibrium operating points. In contrast, in the conventional gain-scheduling approach a family of linear controllers is designed corresponding to the family of series expansion linearizations of the plant relative to the equilibrium operating points only. Nevertheless, the conventional series expansion linearization [Eqs. (4) and (5)] relative to the equilibrium operating point at which δ, q, α, η_Z, and ρ are, respectively, equal to δ_0, q_0, α_0, η_{Z_0}, and ρ_0, can be expressed in state-space form as

$$\begin{bmatrix} \Delta\dot{q} \\ \Delta\dot{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & M_\alpha(\rho_0) \\ 1 & Z_\alpha(\rho_0) \end{bmatrix} \begin{bmatrix} \Delta q \\ \Delta\alpha \end{bmatrix} + \begin{bmatrix} M_\delta(\rho_0) \\ Z_\alpha(\rho_0) \end{bmatrix} \Delta\delta$$

$$\Delta\eta_Z = [0 \quad Z_\alpha(\rho_0)V(\rho_0)] \begin{bmatrix} \Delta q \\ \Delta\alpha \end{bmatrix} + Z_\delta(\rho_0)V(\rho_0)\Delta\delta \tag{36}$$

together with the input, output, and state transformations

$$\Delta\delta = \delta - \delta_0, \qquad q = q_0 + \Delta q, \qquad \alpha = \alpha_0 + \Delta\alpha$$

$$\eta_Z = \eta_{Z_0} + \Delta\eta_Z \tag{37}$$

Clearly, in this example, the members of the conventional series expansion linearization family are closely related to the members of the velocity-based linearization family even though the states, inputs, and outputs are different. In particular, the velocity-based linearization family can be determined directly, by inspection, from the series expansion linearization family provided that there exists an equilibrium operating point corresponding to every value in the range of ρ. This correspondence is certainly not the case in general. For example, the incidence and elevator angles are related at equilibrium operating points but not at nonequilibrium operating points; that is, α_0 is a function of δ, but α in general is not a function of δ and so every pair (α, δ) does not correspond to a pair [$\alpha_0(\delta)$, δ]. Hence, when the incidence and elevator angles are also elements of the scheduling variable, the velocity-based linearization family contains members that do not correspond to any of those of the series expansion linearization family. Nevertheless, in the present missile example the equilibrium operating points are parameterized by the scheduling variable, which is quite typical of flight-control applications where the elements of the scheduling variable are airspeed, dynamic pressure, and Mach number. Consequently, the dynamics at the equilibrium operating points embody the dynamics at every operating point and a gain-scheduled controller designed on the basis of the equilibrium dynamics can be valid even when operating far from equilibrium, for example, during aggressive maneuvering.

In the conventional gain-scheduling approach a family of linear controllers is designed, for example, Eqs. (10) and (11) or (12) and (13), of the form

$$\Delta\dot{x} = A(\rho_0)\Delta x + B(\rho_0)\Delta r \tag{38}$$

$$\Delta\delta = c(\rho_0)\Delta x + d(\rho_0)\Delta r \tag{39}$$

Exploiting the close relationship between the plant series expansion and velocity-based linearization families, the controller velocity-based linearization family, obtained by substituting for the state, input, and output in Eqs. (38) and (39),

$$\dot{\hat{x}} = \hat{w} \tag{40}$$

$$\dot{\hat{w}} = A(\rho_1)\hat{w} + B(\rho_1)\dot{r} \tag{41}$$

$$\dot{\hat{\delta}} = c(\rho_1)\hat{w} + d(\rho_1)\dot{r} \tag{42}$$

is appropriate for the velocity-based gain-scheduling approach. In the conventional gain-scheduling approach the linear controller family [Eqs. (38) and (39)] is typically implemented (ignoring the input, output, and state transformations associated with the perturbation quantities Δr, $\Delta\delta$, Δx) directly as the nonlinear controller

$$\dot{x} = A(\rho)x + B(\rho)r, \qquad \delta = c(\rho)x + d(\rho)r \tag{43}$$

The scheduling variable ρ acts as an implicit input to the nonlinear controller Eq. (43), in addition to the explicit input r. The velocity-based linearization of the nonlinear controller [Eq. (43)], at the operating point (x_1, r_1, ρ_1), is

$$\dot{\hat{x}} = \hat{w} \tag{44}$$

$$\dot{\hat{w}} = A(\rho_1)\hat{w} + B(\rho_1)\dot{r} + \varepsilon_{\hat{w}} \tag{45}$$

$$\dot{\hat{\delta}} = c(\rho_1)\hat{w} + d(\rho_1)\dot{r} + \varepsilon_{\hat{\delta}} \tag{46}$$

where

$$\varepsilon_{\hat{w}} = \nabla_\rho[A(\rho)x]|_{\substack{\rho=\rho_1 \\ x=x_1}}\dot{\rho} + \nabla_\rho[B(\rho)r_1]|_{\substack{\rho=\rho_1 \\ r=r_1}}\dot{\rho}$$

$$\varepsilon_{\hat{\delta}} = \nabla_\rho[c(\rho)x]|_{\substack{\rho=\rho_1 \\ x=x_1}}\dot{\rho} + \nabla_\rho[d(\rho)r]|_{\substack{\rho=\rho_1 \\ r=r_1}}\dot{\rho} \tag{47}$$

Assuming that the rate of variation of ρ is sufficiently slow, $\varepsilon_{\hat{w}}$ and $\varepsilon_{\hat{\delta}}$ are small and can be neglected in Eqs. (44–46); that is, the velocity-based linearization family of the implemented nonlinear controller is, indeed, the required family [Eqs. (40–42)]. Although the transfer function of each member of the linear controller family is unchanged by a nonsingular state transformation, which is different for each member of the family, the dynamics of the corresponding nonlinear controller are changed (see Appendix B). However, the difference in dynamics can be made arbitrarily small by restricting the rate of variation of the scheduling variable ρ. Because this issue is neglected in the conventional gain-scheduling approach, assume that the rate of variation of ρ is sufficiently slow so that the nonlinear controller dynamics are insensitive to the choice of state in the velocity-based linearization family and to the choice of controller realization.

Clearly, the velocity-based analysis provides a rigorous basis for the conventional gain-scheduling approach of designing a family of linear controllers on the basis of the family of series expansion linearizations and employing a direct controller realization of the form [Eq. (43)]. Strong support for the utility of the velocity-based paradigm in the context of the conventional gain-scheduling design approach is provided by its ability to provide an analytic basis for a number of, previously apparently undesirable, aspects of the conventional approach including 1) the practice of neglecting the input, output, and state transformations associated with the linearized descriptions used in analysis and design and 2) the use of frozen scheduling-variable controller linearizations in the design procedure. The latter linearization neglects the variations in the scheduling variable and differs from the series expansion linearization of the nonlinear gain-scheduled controller at the relevant equilibrium operating point. Moreover, the analysis shows that because the dynamics at the equilibrium operating points embody the dynamics at every operating point a gain-scheduled controller designed on the basis of the equilibrium dynamics can be valid, albeit inadvertently, even when operating far from equilibrium. The latter result is quite encouraging because it indicates that the utility of gain-scheduled controllers is considerably greater than suggested by conventional gain-scheduling analysis.

B. Airspeed, Dynamic Pressure, and/or Mach Number not Slowly Varying

The analysis of Sec. IV.A establishes a rigorous basis for the conventional gain-scheduling design approach as typically applied in flight-control applications. However, a number of conditions are required for the conventional approach to be valid. These conditions include the requirements that the short period approximation is accurate, the aerodynamic moment and force are linear with respect to incidence and elevator angle, the equilibrium operating points parameterize the scheduling variable ρ, and, in addition, ρ varies sufficiently slowly so that the nonlinear controller dynamics are insensitive to the choice of state and the velocity-based linearization family of the controller is directly related to the conventional series expansion linearization family. The short period approximation is well established and known to be accurate for a wide variety of flight configurations and operating conditions.[3,12] Similarly, during conventional operation out of stall, the aerodynamic moment and force are frequently approximately linear in incidence and elevator angle (see, for example, Ref. 3, p. 103), and the equilibrium operating points parameterize ρ provided scheduling with respect to incidence is not required. However, the validity of the final assumption that ρ is sufficiently slowly varying is less clear.

It follows from the analysis of Sec. IV.A that when the rate of variation of ρ is sufficiently slow the transfer functions of the closed-loop velocity-based linearizations are the same with controller realizations A and B and the dynamics of the nonlinear controllers are similar. However, whereas the airspeed, dynamic pressure, and Mach number usually vary slowly in comparison to the incidence angle, these scheduling variables are directly related to normal acceleration. Hence, particularly during aggressive maneuvering, the scheduling variable ρ is not a priori slowly varying with respect to the dynamics of the acceleration control loop. This is illustrated by the simulation results presented in Sec. III, where the closed-loop dynamics are significantly different with controller realizations A and B, and inspection of Figs. 2 and 4 reveals that the only difference between the controller realizations lies in the position of the scheduled gain in the acceleration outer loop. The sensitivity of the performance to changes in the controller realization provides, therefore, a direct and immediately relevant indication of the strength of the controller nonlinearity induced by the variation of the scheduling variables. In a similar manner simulation studies indicate that the performance in this example is (unlike the acceleration outer-loop controller) insensitive to the choice of realization of the incidence inner-loop controller.

Because the dependence of the controller dynamics on the choice of realization is a purely nonlinear effect, conventional gain-scheduling analysis provides little insight into this issue. In the present example the requirement is for uniform closed-loop dynamics across the operating envelope, that is, for a dynamic inversion type of controller. Of course, general nonlinear control design methods such as feedback linearization[7,18] or the velocity-based gain-scheduling approach proposed in Ref. 9 can be employed to design a dynamic inversion controller. However, in the present flight-control context a rather simple approach is possible by exploiting the specific structure of the missile and controller dynamics. First, the controller is designed so that the dynamics of the incidence inner loop are sufficiently fast, compared to those of the acceleration outer loop, so that the incidence angle can be assumed to track accurately the demanded incidence angle; that is, α equals $\alpha_d + \varepsilon_\alpha$ where ε_α is small. [Because α is not measured, the inner loop ensures that the estimated incidence angle $\hat{\alpha}$ tracks α_d. It follows from Eq. (48) that $\dot{\alpha} - \dot{\hat{\alpha}} = Z_\alpha(\rho)(\alpha - \hat{\alpha})$, and provided Z_α is negative, $\hat{\alpha}$ is an accurate (after some initial transients) estimate of α.] Second, it follows from Eqs. (32) that

$$\eta_Z = Z_\alpha(\rho)V\alpha + Z_\delta(\rho)V\delta + Z_\varepsilon(\rho)/m \tag{48}$$

where it is assumed, when designing the controller, that Z_δ and Z_ε are sufficiently small so that $Z_\delta(\rho)V\delta + Z_\varepsilon(\rho)/m$ can be treated as a minor disturbance (in fact Z_ε is zero in the present example). In controller realization A the incidence demand satisfies

$$\alpha_d = \left[w_n^2 \big/ Z_\alpha(\rho)V\right]x_4 \tag{49}$$

where x_4 is related to the normal acceleration error by linear dynamics with the transfer function

$$x_4(s) = [1/s(s + 2\varsigma w_n)]\left[\eta_{Z_d}(s) - \eta_Z(s)\right] \tag{50}$$

Hence,

$$\eta_Z = w_n^2 x_4 + Z_\alpha(\rho)V\varepsilon_\alpha + Z_\delta(\rho)V\delta + Z_\varepsilon(\rho)/m \tag{51}$$

and it follows that the normal acceleration dynamics, in pseudo-transfer function form, are

$$\eta_Z(s) = \frac{w_n^2}{s^2 + 2\varsigma w_n s + w_n^2}\eta_{Z_d}(s) + \frac{s^2 + 2\varsigma w_n s}{s^2 + 2\varsigma w_n s + w_n^2}\left[Z_\delta(\rho)V\delta + \frac{Z_\varepsilon(\rho)}{m} + Z_\alpha(\rho)V\varepsilon_\alpha\right] \tag{52}$$

where s denotes the d/dt operator. By assumption, $Z_\delta(\rho)V\delta$, $Z_\varepsilon(\rho)/m$ and $Z_\alpha(\rho)V\varepsilon_\alpha$ are sufficiently small so that they can be neglected in Eq. (52), and the normal acceleration is linearly related to the acceleration demand η_{Z_d} as required. Hence, by adopting an appropriate controller realization, namely, realization A, the conventional gain-scheduling approach can be extended in a straightforward manner to accommodate situations where the airspeed, dynamic pressure, and Mach number are not slowly varying. Conversely, a poor choice of controller realization can lead to unnecessary restrictions on the operating envelope.

The requirement [Eq. (48)] can be relaxed to one so that the acceleration is of the form

$$\eta_Z = z_\alpha(\rho)V\alpha + z_\varepsilon(\rho)V \tag{53}$$

provided that, in the foregoing analysis, $Z_\alpha V$ is replaced by $z_\alpha V$, and $Z_\delta V\delta$ and Z_ε/m are replaced by $z_\varepsilon V$. The condition [Eq. (53)] is quite weak; for example, employing a partial series expansion of Z with respect to α,

$$\eta_Z = \frac{Z}{m} = \frac{1}{m}\left(\frac{\partial Z}{\partial \alpha} + \frac{\partial^2 Z}{\partial \alpha^2}\alpha + \cdots\right)\alpha + \left(\frac{Z}{m}\right)_0 \tag{54}$$

where $(Z/m)_0$ is the acceleration when α is zero and the assumption is made that Z is sufficiently differentiable. The term $(Z/m)_0$ is simply a trim offset that can be treated as a constant disturbance and neglected for the purposes of control design. The infinite series expansion [Eq. (54)] can, of course, be truncated provided the resulting approximation to η_Z is sufficiently accurate over the range of incidence angles associated with the allowable flight envelope.

C. Aerodynamic Force and Moment Nonlinearly Related to Incidence Angle

In the foregoing analysis the assumption is made that the aerodynamic moment and force are linear with respect to the incidence and elevator angles; that is, $\partial M/\partial\alpha$, $\partial M/\partial\delta$, $\partial Z/\partial\alpha$, $\partial Z/\partial\delta$ do not depend on incidence or elevator angle, and scheduling with respect to incidence and elevator angle is not necessary. This is frequently the case during conventional operation out of stall (Ref. 3, p. 103), and scheduling with respect to the instantaneous incidence angle is traditionally avoided[5] (as opposed to, for example, flap scheduling with respect to average incidence obtained using a low-pass filter with very low bandwidth). However, scheduling with respect to the instantaneous incidence angle is likely to be required in, for example, future supermaneuverable aircraft.[5,6]

When $\partial M/\partial\alpha$, $\partial M/\partial\delta$, $\partial Z/\partial\alpha$, $\partial Z/\partial\delta$ depend on the incidence and/or elevator angles, the analysis of Sec. IV.A and B remains valid provided that the scheduling variable now includes the incidence and elevator angles. However, because the incidence and elevator angles almost always vary considerably more rapidly than the airspeed, dynamic pressure, and mach number, the scheduling variable cannot now be assumed to be slowly varying. In particular, it cannot be assumed to be slowly varying with respect to the incidence inner-loop dynamics. Hence, the pitch dynamics can be expected to be sensitive to the choice of realization of both the inner- and outer-loop controllers. Moreover, although the incidence and elevator angles are related at equilibrium operating points, this is not the case at nonequilibrium operating points. Hence, when the incidence and elevator angles are elements of the scheduling variable, the velocity-based linearization family contains members that do not correspond to any of those of the series expansion linearization family, and a gain-scheduled controller designed on the basis of the equilibrium dynamics may not be valid when operating far from equilibrium, for example, during aggressive maneuvering. This observation provides insight into some of the well-known difficulties associated with scheduling on a primary flight variable such as incidence angle, which need not be slowly varying (see, for example, Refs. 5 and 10).

To illustrate the performance degradation that can occur when scheduling with respect to incidence and elevator angle, consider a missile which is similar to that studied in Sec. II except that its aerodynamic moment and force coefficients are described by

$$C_M = -0.3f\alpha - 0.108f\ \sin(3\delta) \tag{55}$$

$$C_L = 0.000103f^3\alpha^3 - 0.00945f^2|\alpha|\alpha + (-0.339 + 0.056\bar{M})f\alpha - 0.034f\delta \tag{56}$$

The response, obtained using nonlinear simulations, with nonlinear controller realization A [with values of the aerodynamic derivatives corresponding to Eqs. (55) and (56)] to a sequence of step demands in normal acceleration is shown in Fig. 5. The missile acceleration is unstable and diverges until the simulation fails because of numeric overflow. The airspeed, dynamic pressure, and Mach number are held constant in the simulation to ensure that the performance degradation is purely a consequence of the nonlinearity of the aerodynamic moment and force with respect to the incidence and elevator angles. Evidently, the conventional gain-scheduling approach is inadequate in this example.

In contrast to the conventional gain-scheduling approach, the velocity-based gain-scheduling methodology does not inherently involve a slow variation requirement and includes information about the dynamics at both equilibrium and nonequilibrium operating points. It therefore has the potential to support the design of an incidence inner-loop controller that can accommodate rapid variations in the scheduling variable and operation far from equilibrium.

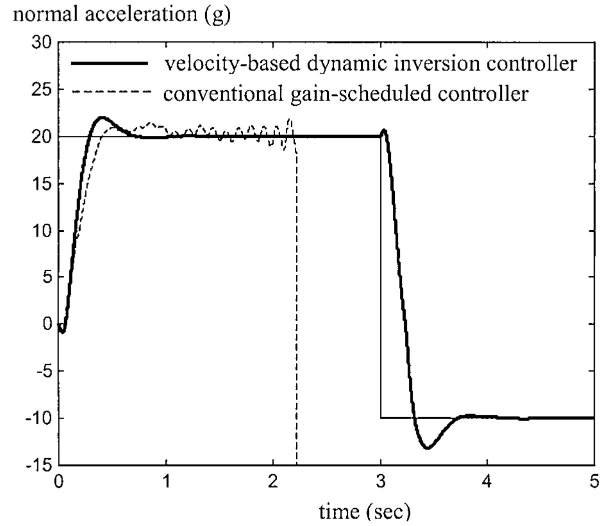

Fig. 5 Performance when aerodynamic moment and force depend nonlinearly on incidence and elevator angles.

The dynamics of the missile are described, in velocity form, by Eqs. (29–31). Assume that the short period approximation is accurate and Z_δ is sufficiently small so that its contribution to the pitch dynamics can be neglected; that is, the pitch dynamics are accurately described by

$$\dot{x} = w, \qquad \dot{w} = \begin{bmatrix} 0 & M_\alpha(\rho) \\ 1 & Z_\alpha(\rho) \end{bmatrix} w + \begin{bmatrix} M_\delta(\rho) \\ 0 \end{bmatrix} \dot{\delta}, \qquad \dot{\alpha} = [0 \quad 1]w \tag{57}$$

where $w = [\dot{q} \ \ \dot{\alpha}]^T$. The members of the velocity-based linearization family associated with the nonlinear dynamics are obtained by simply freezing the scheduling variable ρ in Eqs. (57). The transfer function, of the velocity-based linearization corresponding to the value ρ_1 of the scheduling variable, is

$$\dot{\alpha}(s) = \left\{M_\delta(\rho_1)\Big/\left[s^2 - Z_\alpha(\rho_1)s - M_\alpha(\rho_1)\right]\right\}\dot{\delta}(s) \tag{58}$$

The velocity-based linearizations have relative degree two; that is, the difference between the orders of the numerator and denominator of the transfer function is two. Letting

$$\dot{\delta} = [1/M_\delta(\rho)][(\dddot{u} + a\ddot{u} + b\dot{u})/b] \tag{59}$$

where a and b are positive constants, the pitch dynamics may be reformulated, with input $\dot{u}$, as

$$\dot{\omega} = \begin{bmatrix} 0 & M_\alpha(\rho) \\ 1 & Z_\alpha(\rho) \end{bmatrix} \omega + \begin{bmatrix} \dfrac{M_\alpha(\rho) + b}{b} \\ \dfrac{Z_\alpha(\rho) + a}{b} \end{bmatrix} \dot{u}$$

$$\dot{\alpha} = [0 \quad 1]\omega + (1/b)\dot{u} \tag{60}$$

with

$$\omega = w - \begin{bmatrix} (\ddot{u} + a\dot{u})/b \\ \dot{u}/b \end{bmatrix}$$

The velocity-based linearization of Eq. (60), associated with the value ρ_1 of the scheduling variable, has the transfer function

$$\dot{\alpha}(s) = \frac{1}{b}\left[\frac{s^2 + as + b}{s^2 - Z_\alpha(\rho_1)s - M_\alpha(\rho_1)}\right]\dot{u}(s) \tag{61}$$

and relative degree zero. Adopting a similar approach to that employed in Appendix A to design the linear family for the conventional gain-scheduled controller, let the members of the controller velocity-based linearization family have transfer functions

$$\dot{u}(s) = b\left[\frac{s^2 - Z_\alpha(\rho_1)s - M_\alpha(\rho_1)}{s^2 + as + b}\right]\dot{v}(s) \tag{62}$$

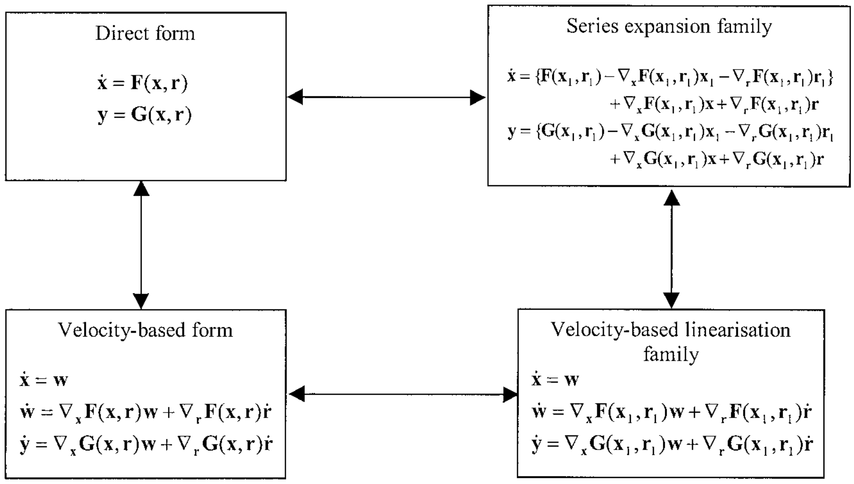

Fig. 6 Alternative representations of a nonlinear system.

which are simply the reciprocal of the transfer functions, [Eq. (61)] of the augmented plant velocity-based linearizations. The cascade combination of each controller velocity-based linearization with the corresponding plant velocity-based linearization has unity transfer function. The velocity-based linearizations of the cascade connection of the plant and controller are simply the cascade connection of the appropriate velocity-based linearizations of the plant and controller[8] and so have unity transfer function. Owing to the direct relationship between the velocity form of a nonlinear system and its velocity-based linearization family (Fig. 6), the dynamics of the cascade combination of the nonlinear plant and controller have linear dynamics with unity transfer function provided that the rate of variation of the scheduling variable $\boldsymbol{\rho}$ is sufficiently slow so that the dynamics are insensitive to the choice of controller state.

The slow variation requirement can be relaxed when the controller state is selected appropriately.[9] In the present case an appropriate choice is

$$\dot{\boldsymbol{x}}^i = \boldsymbol{w}^i, \qquad \dot{\boldsymbol{w}}^i = \begin{bmatrix} 0 & -b \\ 1 & -a \end{bmatrix} \boldsymbol{w}^i + \begin{bmatrix} -M_\alpha(\boldsymbol{\rho}) - b \\ -Z_\alpha(\boldsymbol{\rho}) - a \end{bmatrix} \dot{v}$$

$$\dot{u} = [0 \quad b]\boldsymbol{w}^i + b\dot{v} \tag{63}$$

The transfer function of the velocity-based linearizations of Eqs. (63) associated with operating points at which $\boldsymbol{\rho}$ equals $\boldsymbol{\rho}_1$ is Eq. (62) and when Eqs. (63) are connected in cascade with Eqs. (60), the incidence angle satisfies

$$\dot{\alpha} = [0 \quad 1]z + \dot{v} \tag{64}$$

where $z = \boldsymbol{\omega} + \boldsymbol{w}^i$ and

$$\dot{z} = \begin{bmatrix} 0 & M_\alpha(\boldsymbol{\rho}) \\ 1 & Z_\alpha(\boldsymbol{\rho}) \end{bmatrix} z \tag{65}$$

To ensure internal stability of the cascade combination, assume that the controller dynamics [Eq. (63)] are stable. Provided the unforced dynamics [Eq. (65)] are also stable, z decays to zero, and the incidence angle is related (after some initial transients) to v by linear dynamics with unity transfer function. This approach to dynamic inversion is a direct generalization to nonlinear systems of linear pole-zero cancellation (quite distinct from feedback linearization). It is discussed in detail in Ref. 9. Stability of Eqs. (63) and (65) reduces, in the purely linear case, to the requirement that the transfer function of the plant and its inverse are both stable, thereby avoiding unstable pole-zero cancellation. The stability requirement is not inherent to the velocity-based gain-scheduling approach but rather is a feature of the pole-zero cancellation approach adopted in this example to design the linear controller family on which the gain-scheduled controller is based.

Letting the input to the dynamic inverse controller [Eqs. (63)], be

$$\dot{v}(s) = [K_1/(s + K_2)][\alpha_d(s) - \alpha(s)] \tag{66}$$

the closed-loop pitch dynamics are linear with

$$\alpha(s) = \left[K_1 \Big/ \left(s^2 + K_2 s + K_1\right)\right]\alpha_d(s) \tag{67}$$

as required. Of course, because a measurement of α is not available, the estimated incidence $\hat{\alpha}$ is employed when implementing Eq. (66). It follows from Eq. (53) that the solution to

$$\dot{\hat{\alpha}} = z_\alpha(\boldsymbol{\rho})\hat{\alpha} + z_\varepsilon(\boldsymbol{\rho}) \tag{68}$$

is an accurate estimate of α (after some initial transients that decay to zero provided z_α is negative).

Because u is an internal controller state, $\dot{u}$ and $\ddot{u}$ [required in Eq. (59)] can be readily derived, without numerical differentiation, from Eqs. (63) and (66); namely,

$$\dot{u} = [0 \quad b]\boldsymbol{w}^i + b\dot{v} \tag{69}$$

$$\ddot{u} = b[1 \quad -a]\boldsymbol{w}^i - [Z_\alpha(\boldsymbol{\rho}) + a]\dot{v} + b\{-K_2\dot{v} + K_1[\alpha_d(s) - \alpha(s)]\} \tag{70}$$

However, to derive $\dddot{u}$ requires $\dot{\alpha}_d$, $\dot{\alpha}$, and $\mathrm{d}Z_\alpha(\boldsymbol{\rho})/\mathrm{d}t$, which may not be directly available. Alternatively, therefore, assume that the nonlinear mapping $M(\delta, \boldsymbol{\rho}_m)$, relating the aerodynamic moment to the elevator angle δ and scheduling variable $\boldsymbol{\rho}_m$ (which may depend on the airspeed, incidence angle, etc., but not on δ), is invertible in the sense that there exists an elevator angle $\delta = M^{-1}(M, \boldsymbol{\rho}_m)$, corresponding to every pair $(M, \boldsymbol{\rho}_m)$. This invertibility condition is rather weak and simply requires that the aerodynamic moment is controllable throughout the flight envelope. Letting

$$\delta = M^{-1}\{I_{yy}[\hat{M}_\alpha \alpha + (\ddot{u} + a\dot{u} + bu)/b], \boldsymbol{\rho}_m\} \tag{71}$$

the pitch dynamics can once again be reformulated as Eqs. (60), with $M_\alpha(\boldsymbol{\rho})$ replaced by the constant $\hat{M}_\alpha$. The input transformation [Eq. (71)] does not require the calculation of $\dddot{u}$.

There is no slow variation condition associated with the foregoing inner-loop controller (other than that implied by the initial modeling assumption stating that the short period approximation is accurate). When the velocity-based inner-loop controller is employed and the outer-acceleration-loop controller is as in realization A, the response obtained in the foregoing missile example is shown in Fig. 5. In contrast to the conventional gain-scheduled controller, the velocity-based controller attains the specified performance of a uniform normal acceleration step response over the flight envelope with rise time of around 0.3 s and overshoot less than 25%. This is achieved while retaining the divide and conquer approach and continuity with linear design methods, which are important features of the conventional gain-scheduling approach. Moreover, although not pursued further here, the velocity-based gain-scheduling approach is not confined to the design of dynamic inversion controllers: because a velocity-based linearization is associated with every operating point, the controller can instead be designed such that the closed-loop velocity-based linearizations vary, in some appropriate manner, across the operating envelope.[8]

D. Short Period Approximation Inaccurate

The foregoing sections employ progressively weaker assumptions about the characteristics of the missile dynamics, but the fundamental assumption remaining in Sec. IV.C is that the short period approximation is accurate. However, this requirement can be readily relaxed by letting

$$\dot{\delta} = \dot{\hat{\delta}} - \varepsilon / M_\delta(\boldsymbol{\rho}) \tag{72}$$

where

$$\varepsilon = \left[\frac{\mathrm{d}(M/I_{yy})}{\mathrm{d}t} - \frac{1}{I_{yy}}\frac{\partial M}{\partial \alpha}\dot{\alpha} - \frac{1}{I_{yy}}\frac{\partial M}{\partial \delta}\dot{\delta}\right] + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}(Z/mV)}{\mathrm{d}t} - \frac{1}{mV}\frac{\partial Z}{\partial \alpha}\dot{\alpha}\right] \tag{73}$$

It follows from Eqs. (29–31) that the pitch dynamics can be reformulated as

$$\dot{\boldsymbol{x}} = \boldsymbol{w}, \qquad \dot{\boldsymbol{w}} = \begin{bmatrix} 0 & M_\alpha(\boldsymbol{\rho}) \\ 1 & Z_\alpha(\boldsymbol{\rho}) \end{bmatrix}\boldsymbol{w} + \begin{bmatrix} M_\delta(\boldsymbol{\rho}) \\ 0 \end{bmatrix}\dot{\hat{\delta}}, \qquad \dot{\alpha} = [0 \quad 1]\boldsymbol{w} \tag{74}$$

provided M_δ is not zero. The pitch dynamics [Eqs. (74)], are in the form required by the analysis of Sec. IV.C, which can be reapplied to determine a suitable nonlinear controller.

Alternatively, consider the case when the nonlinear mapping $M(\delta, \boldsymbol{\rho}_m)$, relating the aerodynamic moment to the elevator angle δ and scheduling variable $\boldsymbol{\rho}_m$ (which does not depend on δ), is invertible in the sense that there exists an elevator angle $\delta = M^{-1}(M, \boldsymbol{\rho}_m)$ corresponding to every pair $(M, \boldsymbol{\rho}_m)$. Let

$$\delta = M^{-1}[I_{yy}(\hat{M}_\alpha \alpha + \hat{\delta} - \varepsilon_Z), \boldsymbol{\rho}_m] \tag{75}$$

where

$$\varepsilon_Z = \frac{\mathrm{d}(Z/mV)}{\mathrm{d}t} - \frac{1}{mV}\frac{\partial Z}{\partial \alpha}\dot{\alpha} \tag{76}$$

The missile pitch dynamics can be reformulated as in Eq. (74) [with $M_\alpha(\boldsymbol{\rho})$ replaced by the constant $\hat{M}_\alpha$ and $M_\delta(\boldsymbol{\rho})$ unity]. The expression for ε_Z is somewhat simpler than that for ε and, in particular, involves only the first time derivative of Z/mV.

It is evident from the foregoing that, in comparison to Sec. IV.C, only a relatively small increase in mathematical complexity is required to relax the short period requirement. However, measurements/estimates of the time derivatives of airspeed, dynamic pressure, and Mach number are typically required, which can, in practice, be difficult to obtain reliably. These measurements/estimates are also necessary in other control approaches, such as feedback linearization (see, for example, Ref. 7), and appear to be unavoidable in the general case when the requirement is for uniform dynamics across the flight envelope.

V. Conclusions

Gain scheduling is widely and successfully employed in flight-control applications, where high performance has to be achieved over a broad operating envelope. Nevertheless, conventional theoretical techniques for analyzing the dynamics of gain-scheduled systems are poorly developed and provide little support for the gain-scheduling design approach. Moreover, the suitability of the conventional gain-scheduling approach for designing controllers, which can accommodate, for example, aggressive maneuvering far from equilibrium and operation at high angles of attack/poststall, is unclear. In particular, the sensitivity of gain-scheduled controllers to the choice of nonlinear realization when the scheduling is not sufficiently slowly varying and the difficulties associated with employing incidence angle as a scheduling variable is well known. There is, consequently, interest in the literature in alternative nonlinear control design approaches such as dynamic inversion. However, owing to the substantial body of experience that has been accumulated with gain-scheduling methods both with regard to meeting performance requirements and also such practical issues as safety certification, there is a strong incentive to retain the gain-scheduling approach while resolving the foregoing difficulties. The recently developed velocity-based analysis framework associates a linear system with every operating point of a nonlinear system, not just the equilibrium operating points, and so is not confined to near equilibrium operation and does not inherently involve a slow variation requirement. Furthermore, it provides a natural and unified framework for gain-scheduling analysis and design that addresses many of the shortcomings of conventional gain-scheduling analysis and design while retaining the continuity with linear methods, which is the principle feature of the conventional approach.

In this paper the conventional gain-scheduling design approach to controlling the longitudinal dynamics of a missile is shown to be equivalent to the velocity-based gain-scheduling approach provided that 1) the short period approximation is accurate; 2) the aerodynamic moment and force are linear with respect to incidence and elevator angle; 3) the equilibrium operating points parameterize the scheduling variable; and 4) the scheduling variable varies sufficiently slowly that the nonlinear controller dynamics are insensitive to the choice of realization and the velocity-based linearization family of the controller is directly related to the conventional series expansion linearization family.

The analysis thereby establishes a rigorous basis for the conventional gain-scheduling design approach. Strong support for the utility of the velocity-based paradigm in the context of the conventional gain-scheduling design approach is provided by its ability to provide an analytic basis for a number of, previously apparently undesirable, aspects of the conventional approach including 1) the practice of neglecting the input, output, and state transformations associated with the linearized descriptions used in analysis and design and 2) the use of frozen scheduling-variable controller linearizations in the design procedure. The latter linearization neglects the variations in the scheduling variable and differs from the series expansion linearization of the nonlinear gain-scheduled controller at the relevant equilibrium operating point. In addition, although the conventional gain-scheduling design approach is based on the linearizations of the plant about equilibrium operating points, it is shown that, under the foregoing conditions, the resulting gain-scheduled controller is valid, albeit inadvertently, even when operating far from equilibrium. The latter result is quite encouraging because it indicates that the utility of gain-scheduled controllers is considerably greater than suggested by conventional gain-scheduling analysis.

In addition to providing insight into the conventional gain-scheduling approach, the velocity-based analysis and design framework is employed to design gain-scheduled controllers, which progressively relax the restrictions (1–4) required by the conventional approach while retaining, as far as possible, the continuity with linear design methods of conventional gain scheduling. In particular, rigorous insight is thereby provided into issues such as the sensitivity of gain-scheduled controllers to the choice of nonlinear realization and the difficulties associated with employing the incidence and elevator angles as scheduling variables. This insight is exploited to develop gain-scheduled control designs that resolve these issues, and the effectiveness of these designs is illustrated by a number of simulation trials.

Although illustrated with reference to a simple missile example, the analysis and design techniques developed are, of course, also relevant to other flight-control applications.

Appendix A: Missile Gain-Scheduled Controller

The requirement is to design a family of local linear controllers for the family of linear plants [Eq. (9)]. A natural choice of controller configuration is a cascaded inner-outer-loop arrangement that employs an attitude inner loop and a normal acceleration outer loop. With this approach the acceleration outer loop supplies an angle of incidence demand to the attitude inner loop based on the normal acceleration demanded. First, consider the design of the inner-loop controller. It follows from the short period dynamics [Eq. (9)] that, locally to the equilibrium operating point π_0, the transfer function relating incidence angle to elevator angle is

$$\Delta\alpha(s) = \frac{M_\delta(\rho_0)}{s^2 - Z_\alpha(\rho_0)s - M_\alpha(\rho_0)}\Delta\delta(s) \tag{A1}$$

where $\Delta\alpha(s)$ and $\Delta\delta(s)$ are, respectively, the Laplace transforms of $\Delta\alpha$ and $\Delta\delta$. Selecting

$$\Delta\delta(s) = \frac{K_1}{M_\delta(\rho_0)}\frac{[s^2 - Z_\alpha(\rho_0)s - M_\alpha(\rho_0)]}{s(s+K_2)}[\Delta\alpha_d(s) - \Delta\alpha(s)] \tag{A2}$$

where $\Delta\alpha_d$ is the incidence demand from the outer-loop acceleration controller, the closed-loop transfer function of the incidence loop is

$$\Delta\alpha(s) = \left[K_1\big/\left(s^2 + K_2 s + K_1\right)\right]\Delta\alpha_d(s) \tag{A3}$$

Of course, an angle of incidence measurement is not available in the present example. However, it can be estimated from the pitch-rate measurement using

$$\Delta\dot{\hat{\alpha}} = Z_\alpha(\rho_0)\Delta\hat{\alpha} + Z_\delta(\rho_0)\Delta\delta + \Delta q \tag{A4}$$

where $\Delta\hat{\alpha}$ is the estimate of $\Delta\alpha$. The controller parameters K_1 and K_2 are selected to ensure that the inner-loop dynamics have a satisfactory natural frequency and damping while respecting the limitations of the actuator and the need to avoid exciting elastic modes of the airframe; in the present example values for these parameters are selected that correspond to a natural frequency of 50 rad/s and damping factor of 0.7.

Second, the design of the outer-loop normal acceleration controller is addressed. A normal acceleration step response with rise time (to 95% of final value) of around 0.3 s corresponds to a natural frequency of around 10 rad/s and a damping factor of 0.7. Because the inner-loop bandwidth of 50 rad/s is large compared to the required bandwidth of the acceleration loop, the design of these loops is effectively decoupled; that is, the dynamics of the inner loop may be neglected when designing the outer-loop controller. This timescale separation is quite natural and, indeed, inherent to the missile configuration because a change in acceleration is initiated by a change in incidence angle, and so incidence must always change more rapidly than normal acceleration. Hence, the assumption can be made in this context that $\Delta\alpha$ equals $\Delta\alpha_d$. Because the sign of the coefficient of δ in C_L is negative, the normal acceleration in response to a change in the elevator angle δ is initially in the opposite direction; that is, the normal acceleration response is nonminimum phase. However, the right-half plane zero associated with the nonminimum phase character of the acceleration response lies well outside the frequency range of interest and can, therefore, also be neglected when designing the outer-loop controller. The gain $Z_\alpha V$, relating changes in the angle of incidence to changes in normal acceleration, varies strongly with the flight condition, and in order to accommodate this variation, the gain of the outer-loop controller must also be varied. An appropriate outer-loop controller transfer function is, therefore,

$$\Delta\alpha_d(s) = [1/Z_\alpha(\rho_0)V(\rho_0)]\left[w_n^2\big/s(s+2\varsigma w_n)\right]\left(\Delta\eta_{Z_d} - \Delta\eta_Z\right) \tag{A5}$$

where $\Delta\eta_{Z_d}$ is the perturbation in the demanded normal acceleration and $w_n = 10$ rad/s and $\zeta = 0.7$. The corresponding closed-loop dynamics obtained when this outer-loop controller is combined with the linearized plant dynamics [Eq. (5)] (neglecting the inner-loop dynamics) are

$$\Delta\eta_Z(s) = \frac{w_n^2}{s^2 + 2\varsigma w_n s + w_n^2}\Delta\eta_{Z_d}(s) + \frac{s^2 + 2\varsigma w_n s}{s^2 + 2\varsigma w_n s + w_n^2}Z_\delta(\rho_0)V(\rho_0)\Delta\delta \tag{A6}$$

Evidently, the term $Z_\delta(\rho_0)V(\rho_0)\Delta\delta$ in Eq. (A6) is attenuated over the control bandwidth, and, consequently, the normal acceleration of the missile $\Delta\eta_Z$ is essentially related to the demanded acceleration $\Delta\eta_{Z_d}$ by second-order dynamics with natural frequency 10 rad/s and damping of 0.7, as required.

A family of linear controller transfer function designs is defined by Eqs. (A2), (A4), and (A5), which corresponds to the family of plant linearizations [Eq. (9)], and is parameterized by $\boldsymbol{\rho}$.

Appendix B: Sensitivity to Choice of Controller Realization

The nonlinear system

$$\dot{x} = w \tag{B1}$$

$$\dot{w} = \nabla_x F(x, r)w + \nabla_r F(x, r)\dot{r} \tag{B2}$$

$$\dot{y} = \nabla_x G(x, r)w + \nabla_r G(x, r)\dot{r} \tag{B3}$$

has, at the operating point (x_1, r_1), the velocity-based linearization

$$\dot{\hat{x}} = \hat{w} \tag{B4}$$

$$\dot{\hat{w}} = \nabla_x F(x_1, r_1)\hat{w} + \nabla_r F(x_1, r_1)\dot{r} \tag{B5}$$

$$\dot{\hat{y}} = \nabla_x G(x_1, r_1)\hat{w} + \nabla_r G(x_1, r_1)\dot{r} \tag{B6}$$

Of course, the dynamics of a linear system are invariant under a nonsingular state transformation. Consider, therefore, the nonlinear system

$$\dot{\chi} = T(\chi, r)\omega \tag{B7}$$

$$\dot{\omega} = T^{-1}(\chi, r)\nabla_x F(\chi, r)T(\chi, r)\omega + T^{-1}(\chi, r)\nabla_r F(\chi, r)\dot{r} \tag{B8}$$

$$\dot{\nu} = \nabla_x G(\chi, r)T(\chi, r)\omega + \nabla_r G(\chi, r)\dot{r} \tag{B9}$$

for which the velocity-based linearization, at the operating point (x_1, r_1), is

$$\dot{\hat{\chi}} = T(x_1, r_1)\hat{\omega} \tag{B10}$$

$$\dot{\hat{\omega}} = T^{-1}(x_1, r_1)\nabla_x F(x_1, r_1)T(x_1, r_1)\hat{\omega} + T^{-1}(x_1, r_1)\nabla_r F(x_1, r_1)\dot{r} \tag{B11}$$

$$\dot{\hat{\nu}} = \nabla_x G(x_1, r_1)T(x_1, r_1)\hat{\omega} + \nabla_r G(x_1, r_1)\dot{r} \tag{B12}$$

where χ, ω, $\hat{\chi}$, $\hat{\omega} \in \Re^n$ and $T(\cdot,\cdot)$ is a uniformly bounded nonsingular matrix, which is differentiable with uniformly bounded derivatives. The velocity-based linearizations [Eqs. (B4–B6) and (B10–B12)] are related by the nonsingular transformation

$$\hat{\omega} = T^{-1}(x_1, r_1)\hat{w}, \qquad \hat{\chi} = \hat{x}, \qquad \hat{\nu} = \hat{y} \tag{B13}$$

and so are dynamically equivalent. A similar situation applies to the velocity-based linearizations at other operating points. However, letting

$$\omega = T^{-1}(\chi, r)z \tag{B14}$$

the nonlinear system [Eqs. (B7–B9)] can be reformulated as

$$\dot{\chi} = z \tag{B15}$$

$$\dot{z} = \nabla_x F(\chi, r)z + \nabla_r F(\chi, r)\dot{r} + \varepsilon \tag{B16}$$

$$\dot{\nu} = \nabla_x G(\chi, r)z + \nabla_r G(\chi, r)\dot{r} \tag{B17}$$

where $\varepsilon = \dot{T}(\chi, r)T^{-1}(\chi, r)z$. Hence, despite the dynamic equivalence of the members of the velocity-based linearization families, it is evident that the dynamics of the nonlinear systems [Eqs. (B1–B3) and (B7–B9)] are not equivalent. The difference between the dynamics is embodied by the perturbation term ε and arises from the variation of the state transformation [Eq. (B13)] with the operating point. Nevertheless, the difference between the solutions to the nonlinear systems [Eqs. (B1–B3) and (B15–B17)] over any finite time interval is arbitrarily small provided the magnitude of ε is sufficiently small (see, for example, Ref. 19, theorem 2.5).

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References

[1]Astrom, K. J., and Wittenmark, B., *Adaptive Control*, Addison Wesley Longman, Reading, MA, 1989.

[2]Hyde, R. A., and Glover, K., "The Application of Scheduled H_∞ Controllers to a VSTOL Aircraft," *IEEE Transactions on Automatic Control*, Vol. 38, No. 7, 1993, pp. 1021–1039.

[3]Mclean, D., *Automatic Flight Control Systems*, Prentice–Hall, Upper Saddle River, NJ, 1990.

[4]Leith, D. J., and Leithead, W. E., "Gain-Scheduled & Nonlinear & Systems: Dynamic Analysis by Velocity-Based Linearization Families," *International Journal of Control*, Vol. 70, 1998, pp. 289–317.

[5]Snell, S. A., Enns, D. F., and Garrard, W. L., "Nonlinear Control of a Super-Manoeuvrable Aircraft," *Proceedings of the AIAA Guidance, Navigation and Control Conference*, AIAA, Washington, DC, 1989, pp. 519–531.

[6]Garrard, W. L., Enns, D. F., and Snell, S. A., "Nonlinear Feedback Control of Highly Manoeuvrable Aircraft," *International Journal of Control*, Vol. 56, No. 4, 1992, pp. 799–812.

[7]Meyer, G., Su, R., and Hunt, L. R., "Application of Nonlinear Transformations to Automatic Flight Control," *Automatica*, Vol. 20, No. 1, 1984, pp. 103–107.

[8]Leith, D. J., and Leithead, W. E., "Gain-Scheduled Controller Design: An Analytic Framework Directly Incorporating Non-Equilibrium Plant Dynamics," *International Journal of Control*, Vol. 70, No. 2, 1998, pp. 249–269.

[9]Leith, D. J., and Leithead, W. E., "Input-Output Linearisation by Velocity-Based Gain-Scheduling," *International Journal of Control*, Vol. 72, No. 3, 1999, pp. 229–246.

[10]Nichols, R. A., Reichert, R. T., and Rugh, W. J., "Gain-Scheduling H-Infinity Controllers: A Flight Control Example," *IEEE Transactions on Control Systems Technology*, Vol. 1, No. 2, 1993, pp. 69–78.

[11]Shamma, J. S., and Cloutier, J. R., "A Linear Parameter-Varying Approach to Gain-Scheduled Missile Autopilot Design," *Proceedings of the American Control Conference*, 1992, pp. 1317–1321.

[12]Blakelock, J., *Automatic Control of Aircraft and Missiles*, Wiley, New York, 1996.

[13]Lawrence, D. A., and Rugh, W. J., "Gain Scheduling Dynamic Linear Controllers for a Nonlinear Plant," *Automatica*, Vol. 31, No. 3, 1995, pp. 81–390.

[14]Leith, D. J., and Leithead, W. E., "Appropriate Realisation of Gain Scheduled Controllers with Application to Wind Turbine Regulation," *International Journal of Control*, Vol. 65, No. 2, 1997, pp. 223–248.

[15]Leith, D. J., and Leithead, W. E., "Appropriate Realization of MIMO Gain Scheduled Controllers," *International Journal of Control*, Vol. 70, No. 1, 1998, pp. 13–50.

[16]Leith, D. J., and Leithead, W. E., "Comments on Gain Scheduling Dynamic Linear Controllers for a Nonlinear Plant," *Automatica*, Vol. 34, No. 8, 1998, pp. 1041–1043.

[17]Leith, D. J., and Leithead, W. E., "Further Comments on 'Gain-Scheduling Dynamic Linear Controllers for a Nonlinear Plant," *Automatica*, Vol. 36, No. 2, 1999, pp. 173,174.

[18]Isidori, A., *Nonlinear Control Systems*, Springer–Verlag, London, 1995.

[19]Khalil, H. K., *Nonlinear Systems*, Macmillan, New York, 1992.